



# Link complexes of subspace arrangements

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## Abstract

Given a simplicial hyperplane arrangement  $\mathcal{H}$  and a subspace arrangement  $\mathcal{A}$  embedded in  $\mathcal{H}$ , we define a simplicial complex  $\Delta_{\mathcal{A}, \mathcal{H}}$  as the subdivision of the link of  $\mathcal{A}$  induced by  $\mathcal{H}$ . In particular, this generalizes Steingrímsson's coloring complex of a graph.

We do the following:

- (1) When  $\mathcal{A}$  is a hyperplane arrangement,  $\Delta_{\mathcal{A}, \mathcal{H}}$  is shown to be shellable. As a special case, we answer affirmatively a question of Steingrímsson on coloring complexes.
- (2) For  $\mathcal{H}$  a Coxeter arrangement of type  $A$  or  $B$  we obtain a close connection between the Hilbert series of the Stanley–Reisner ring of  $\Delta_{\mathcal{A}, \mathcal{H}}$  and the characteristic polynomial of  $\mathcal{A}$ . This extends results of Steingrímsson and provides an interpretation of chromatic polynomials of hypergraphs and signed graphs in terms of Hilbert polynomials.

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## 1. Introduction

In [10], Steingrímsson introduced the *coloring complex*  $\Delta_G$ . This is a simplicial complex associated with a graph  $G$ . The Hilbert polynomial of its Stanley–Reisner ring  $k[\Delta_G]$  is closely related to the chromatic polynomial  $P_G(x)$  in a way that is made precise in Section 5.

Answering a question of Steingrímsson, Jonsson [7] proved that  $\Delta_G$  is a Cohen–Macaulay complex by showing that it is constructible. In particular,  $\Delta_G$  being Cohen–Macaulay imposes restrictions on the Hilbert polynomial of  $k[\Delta_G]$ , and hence on  $P_G(x)$ .

Since  $\Delta_G$  is a Cohen–Macaulay complex, a natural question, asked already in [10], is whether it is shellable — a stronger property than constructibility.

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In [10],  $\Delta_G$  was defined in a combinatorially very explicit way. Another way to view  $\Delta_G$  is, however, as a simplicial decomposition of the link (i.e. intersection with the unit sphere) of the *graphical hyperplane arrangement* associated with  $G$ . In this guise,  $\Delta_G$  appeared in work of Herzog et al. [6]. Adopting this point of view, one may define a similar complex  $\Delta_{\mathcal{A}, \mathcal{H}}$  for any subspace arrangement  $\mathcal{A}$ , as long as it has an embedding in a simplicial hyperplane arrangement  $\mathcal{H}$ .

This paper has two goals. The first is addressed in Section 4 where we show that  $\Delta_{\mathcal{A}, \mathcal{H}}$  is shellable whenever  $\mathcal{A}$  consists of hyperplanes. In particular, this proves that the coloring complexes are shellable.

The chromatic polynomial of  $G$  is essentially the characteristic polynomial of the corresponding graphical hyperplane arrangement. Bearing this in mind, one may hope to extend the aforementioned connection between the Hilbert polynomial of  $k[\Delta_G]$  and  $P_G(x)$  to more general complexes  $\Delta_{\mathcal{A}, \mathcal{H}}$ . Achieved in Section 5, our second goal is to carry out this extension whenever  $\mathcal{H}$  is a Coxeter arrangement of type  $A$  or  $B$ . When  $\mathcal{A}$  consists of hyperplanes and  $\mathcal{H}$  is of type  $A$ , Steingrímsson's result is recovered.

We define the complexes  $\Delta_{\mathcal{A}, \mathcal{H}}$  in Section 3 after reviewing some necessary background in the next section.

## 2. Preliminaries

### 2.1. Subspace arrangements and characteristic polynomials

By the term *subspace arrangement* we mean a finite collection  $\mathcal{A} = \{A_1, \dots, A_t\}$  of linear subspaces, none of which contains another, of some ambient vector space. In our case, the ambient space will always be  $\mathbb{R}^n$  for some  $n$ . With  $\mathcal{A}$  we associate the *intersection lattice*  $L_{\mathcal{A}}$  which consists of all intersections of subspaces in  $\mathcal{A}$  ordered by reverse inclusion. (We emphasize the fact that  $\mathcal{A}$  contains no strictly affine subspaces; in particular this implies that  $L_{\mathcal{A}}$  is indeed a lattice.)

An important invariant of the arrangement  $\mathcal{A}$  is its *characteristic polynomial*

$$\chi(\mathcal{A}; x) = \sum_{Y \in L_{\mathcal{A}}} \mu(\hat{0}, Y) x^{\dim(Y)},$$

where  $\mu$  is the Möbius function of  $L_{\mathcal{A}}$  and  $\hat{0} = \mathbb{R}^n$  is the smallest element in  $L_{\mathcal{A}}$ .

Given a subspace  $A \in \mathcal{A}$ , we define two new arrangements, namely the *deletion*

$$\mathcal{A} \setminus A = \mathcal{A} \setminus \{A\}$$

and the *restriction*

$$\mathcal{A}/A = \max\{A \cap B \mid B \in \mathcal{A} \setminus A\},$$

where  $\max \mathcal{S}$  denotes the collection of inclusion-maximal members of a set family  $\mathcal{S}$ . Another way to think of  $\mathcal{A}/A$  is as the set of elements covering  $A$  in  $L_{\mathcal{A}}$ . In this way, we may extend the definition of  $\mathcal{A}/A$  to arbitrary  $A \in L_{\mathcal{A}}$ . We consider  $\mathcal{A} \setminus A$  to be an arrangement in  $\mathbb{R}^n$ , whereas  $\mathcal{A}/A$  is an arrangement in  $A$ .

When  $\mathcal{A}$  is a hyperplane arrangement, the next result is standard. We expect the general case to be known, too, although we have been unable to find it in the literature.

**Theorem 2.1** (*Deletion–Restriction*). *For a subspace arrangement  $\mathcal{A}$  and any subspace  $A \in \mathcal{A}$ , we have*

$$\chi(\mathcal{A}; x) = \chi(\mathcal{A} \setminus A; x) - \chi(\mathcal{A}/A; x).$$

**Proof.** Choose  $Y \in L_{\mathcal{A}}$ . We claim that

$$\mu_{\mathcal{A}}(\hat{0}, Y) = \begin{cases} \mu_{\mathcal{A} \setminus A}(\hat{0}, Y) - \mu_{\mathcal{A}}(A, Y) & \text{if } Y \in L_{\mathcal{A} \setminus A}, \\ -\mu_{\mathcal{A}}(A, Y) & \text{otherwise} \end{cases}$$

where  $\mu_{\mathcal{A}}$  denotes the Möbius function of  $L_{\mathcal{A}}$  which we think of as a function  $L_{\mathcal{A}} \times L_{\mathcal{A}} \rightarrow \mathbb{Z}$  with  $S \not\leq T \Rightarrow \mu_{\mathcal{A}}(S, T) = 0$  (and similarly for  $\mathcal{A} \setminus A$ ).

The claim is true if  $Y = \hat{0} = \mathbb{R}^n$ , so assume it has been verified for all  $Z < Y$  in  $L_{\mathcal{A}}$ . If  $Y \in L_{\mathcal{A} \setminus A}$  we obtain

$$\begin{aligned} \mu_{\mathcal{A}}(\hat{0}, Y) &= - \sum_{\hat{0} \leq Z < Y} \mu_{\mathcal{A}}(\hat{0}, Z) = - \sum_{\substack{\hat{0} \leq Z < Y \\ Z \in L_{\mathcal{A} \setminus A}}} \mu_{\mathcal{A} \setminus A}(\hat{0}, Z) + \sum_{A \leq Z < Y} \mu_{\mathcal{A}}(A, Z) \\ &= \mu_{\mathcal{A} \setminus A}(\hat{0}, Y) - \mu_{\mathcal{A}}(A, Y), \end{aligned}$$

as desired. If, on the other hand,  $Y \notin L_{\mathcal{A} \setminus A}$ , then there is a unique largest element in  $L_{\mathcal{A} \setminus A}$  which is below  $Y$  in  $L_{\mathcal{A}}$ , namely the join of all atoms (weakly) below  $Y$  except  $A$ ; call this element  $W$ . If  $W = \hat{0}$ , then  $Y = A$  and we are done. Otherwise,

$$\begin{aligned} \mu_{\mathcal{A}}(\hat{0}, Y) &= - \sum_{\hat{0} \leq Z < Y} \mu_{\mathcal{A}}(\hat{0}, Z) = - \sum_{\substack{\hat{0} \leq Z \leq W \\ Z \in L_{\mathcal{A} \setminus A}}} \mu_{\mathcal{A} \setminus A}(\hat{0}, Z) + \sum_{A \leq Z < Y} \mu_{\mathcal{A}}(A, Z) \\ &= \sum_{A \leq Z < Y} \mu_{\mathcal{A}}(A, Z) = -\mu_{\mathcal{A}}(A, Y), \end{aligned}$$

establishing the claim.

We conclude that

$$\chi(\mathcal{A}; x) = \sum_{Y \in L_{\mathcal{A} \setminus A}} \mu_{\mathcal{A} \setminus A}(\hat{0}, Y) x^{\dim(Y)} - \sum_{Y \geq A} \mu_{\mathcal{A}}(A, Y) x^{\dim(Y)}.$$

Not every  $Y$  in the last sum belongs to  $L_{\mathcal{A}/A}$  in general; the latter is join-generated by the elements covering  $A$  in  $L_{\mathcal{A}}$ . However, it follows from Rota's Crosscut theorem [8] that for every  $Y \geq A$  in  $L_{\mathcal{A}}$ ,

$$\mu_{\mathcal{A}}(A, Y) = \begin{cases} \mu_{\mathcal{A}/A}(A, Y) & \text{if } Y \in L_{\mathcal{A}/A}, \\ 0 & \text{otherwise.} \end{cases}$$

Thus,

$$\sum_{Y \geq A} \mu_{\mathcal{A}}(A, Y) x^{\dim(Y)} = \chi(\mathcal{A}/A; x),$$

and the theorem follows.  $\square$

Two (families of) hyperplane arrangements are of particular importance to us. The first is the *braid arrangement*  $\mathcal{S}_n$ . This is an arrangement whose ambient space is  $\{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_1 + \dots + x_n = 0\} \cong \mathbb{R}^{n-1}$ . The  $\binom{n}{2}$  hyperplanes in  $\mathcal{S}_n$  are given by the equations  $x_i = x_j$  for all  $1 \leq i < j \leq n$ .

The braid arrangement is the set of reflecting hyperplanes of a Weyl group of type  $A$ . Considering type  $B$  instead, we find our second important family of arrangements. Explicitly,  $\mathcal{B}_n$  is the arrangement of the  $n^2$  hyperplanes in  $\mathbb{R}^n$  that are given by the equations  $x_i = \tau x_j$  for all  $1 \leq i < j \leq n$ ,  $\tau \in \{-1, 1\}$ , and  $x_i = 0$  for all  $i \in [n] = \{1, \dots, n\}$ .

## 2.2. Stanley–Reisner rings and $h$ -polynomials

Let  $\Delta$  be a simplicial complex on the vertex set  $[n]$ . Regarding the vertices as variables, we want to consider the ring of polynomials that live on  $\Delta$ . To this end, for a field  $k$ , we define the *Stanley–Reisner ideal*  $I_\Delta \subseteq k[x_1, \dots, x_n]$  by

$$I_\Delta = \langle \{x_{i_1} \dots x_{i_t} \mid \{i_1, \dots, i_t\} \not\subseteq \Delta\} \rangle.$$

The quotient ring

$$k[\Delta] = k[x_1, \dots, x_n]/I_\Delta$$

is the *Stanley–Reisner ring* of  $\Delta$ , which is a graded algebra with the standard grading by degree. When speaking of algebraic properties, such as Cohen–Macaulayness, of  $\Delta$  we have the corresponding properties of  $k[\Delta]$  in mind.

Given a simplicial complex  $\Delta$  of dimension  $d - 1$ , its  *$h$ -polynomial* is

$$h(\Delta; x) = \sum_{i=0}^d f_{i-1} (x-1)^{d-i},$$

where  $f_i$  is the number of  $i$ -dimensional simplices in  $\Delta$  (including  $f_{-1} = 1$  if  $\Delta$  is nonempty). One important feature of the  $h$ -polynomial is that it carries all information needed to compute the Hilbert series of  $k[\Delta]$ . Specifically,

$$\text{Hilb}(k[\Delta]; x) = \frac{\bar{h}(\Delta; x)}{(1-x)^d},$$

where  $\bar{h}$  denotes the reverse  $h$ -polynomial:

$$\bar{h}(\Delta; x) = x^d h\left(\Delta; \frac{1}{x}\right).$$

## 2.3. Shellable complexes

Suppose  $\Delta$  is a *pure* simplicial complex, meaning that all facets (maximal simplices) have the same dimension  $d - 1$ . A *shelling order* for  $\Delta$  is a total ordering  $F_1, \dots, F_t$  of the facets of  $\Delta$  such that  $F_j \cap (\cup_{i < j} F_i)$  is pure of dimension  $d - 2$  for all  $j = 2, \dots, t$ . We say that  $\Delta$  is *shellable* if a shelling order for  $\Delta$  exists.

One good reason to care about shellability is that it implies Cohen–Macaulayness.

## 3. The objects of study

Suppose  $\mathcal{H}$  is a hyperplane arrangement in  $\mathbb{R}^n$  such that  $\cap \mathcal{H} = \{0\}$ . Then,  $\mathcal{H}$  determines a regular cell decomposition  $\Delta_{\mathcal{H}}$  of the unit sphere  $S^{n-1}$ . In short, each point  $p$  on  $S^{n-1}$  has an associated sign vector in  $\{0, -, +\}^{|\mathcal{H}|}$  recording for each hyperplane  $h \in \mathcal{H}$  whether  $p$  is on, or on the negative, or on the positive side of  $h$  (for some choice of orientations of the hyperplanes). A cell in  $\Delta_{\mathcal{H}}$  consists of the set of points with a common sign vector. The face poset of  $\Delta_{\mathcal{H}}$  is the big face lattice of the corresponding oriented matroid; see [2].

If  $\Delta_{\mathcal{H}}$  is a simplicial complex, then  $\mathcal{H}$  is called *simplicial*. A prime example of a simplicial hyperplane arrangement is the collection of reflecting hyperplanes of a finite Coxeter group. In this case,  $\Delta_{\mathcal{H}}$  coincides with the Coxeter complex.

From now on, let  $\mathcal{H}$  be a simplicial hyperplane arrangement.

Consider an antichain  $\mathcal{A}$  in  $L_{\mathcal{H}}$ . We say that the subspace arrangement  $\mathcal{A}$  is *embedded* in  $\mathcal{H}$ . Observe that  $\cup \mathcal{A} \cap S^{n-1}$ , which is known as the *link* of  $\mathcal{A}$ , has the structure of a simplicial subcomplex of  $\Delta_{\mathcal{H}}$ . This subcomplex is the principal object of study in this paper. We denote it  $\Delta_{\mathcal{A}, \mathcal{H}}$ .

**Example 3.1.** A graph  $G = ([n], E)$  determines a *graphical hyperplane arrangement*  $\widehat{G}$  in the  $(n - 1)$ -dimensional subspace of  $\mathbb{R}^n$  given by the equation  $x_1 + \cdots + x_n = 0$ . There is one hyperplane in  $\widehat{G}$  for each edge in  $E$ ; the hyperplane corresponding to the edge  $\{i, j\}$  has the equation  $x_i = x_j$ .

The arrangement  $\widehat{K}_n$  corresponding to the complete graph is nothing but the braid arrangement  $S_n$  which is simplicial. Any graph  $G$  thus determines a simplicial complex  $\Delta_{\widehat{G}, S_n}$ . It coincides with Steingrímsson's coloring complex of  $G$  which was denoted by  $\Delta_G$  in the Introduction. The complex  $\Delta_{\widehat{G}, S_n}$  also appeared under the name  $\Delta_{m, J}$  in [6].

We remark that the homotopy type of the link of  $\mathcal{A}$ , and hence of  $\Delta_{\mathcal{A}, \mathcal{H}}$ , can be computed in terms of the order complexes of lower intervals in  $L_{\mathcal{A}}$  by a formula of Ziegler and Živaljević [13]. When  $\mathcal{A}$  consists of hyperplanes we may simply note that  $\Delta_{\mathcal{A}, \mathcal{H}}$  is homotopy equivalent to the  $(n - 1)$ -sphere with one point removed for each connected region in the complement  $\mathbb{R}^n \setminus \cup \mathcal{A}$ . Denoting by  $R(\mathcal{A})$  the number of such regions,  $\Delta_{\mathcal{A}, \mathcal{H}}$  is thus homotopy equivalent to a wedge of  $R(\mathcal{A}) - 1$  spheres of dimension  $n - 2$  in this case. For the arrangements  $\widehat{G}$  of Example 3.1 it is not difficult to see that  $R(\widehat{G})$  equals the number  $AO(G)$  of acyclic orientations of  $G$ . Thus,  $\Delta_{\widehat{G}, S_n}$  has the homotopy type of a wedge of  $AO(G) - 1$   $(n - 3)$ -spheres [6,7]. In particular, the reduced Euler characteristic of  $\Delta_{\widehat{G}, S_n}$  is  $\pm(AO(G) - 1)$  [10, Theorem 17].

#### 4. Shellability in the hyperplane case

Our goal in this section is to show that  $\Delta_{\mathcal{A}, \mathcal{H}}$  is shellable whenever  $\mathcal{A}$  consists of hyperplanes. Applied to the complexes  $\Delta_{\widehat{G}, S_n}$  of Example 3.1 this answers affirmatively a question of Steingrímsson [10] which was restated in [7]. The key tool is a particular class of shellings of  $\Delta_{\mathcal{H}}$  determined by the *poset of regions* of  $\mathcal{H}$  which we now define.

The complement  $\mathbb{R}^n \setminus \cup \mathcal{H}$  is cut into disjoint open regions by  $\mathcal{H}$ . Restricting to the unit sphere, their closures are the facets of  $\Delta_{\mathcal{H}}$ . Let  $\mathcal{F} = \mathcal{F}(\mathcal{H})$  be the set of such facets. For  $R, R' \in \mathcal{F}$ , say that  $h \in \mathcal{H}$  *separates*  $R$  and  $R'$  if their respective interiors are on different sides of  $h$ .

Choose a *base region*  $B \in \mathcal{F}$  arbitrarily. We have a distance function  $\ell : \mathcal{F} \rightarrow \mathbb{N}$  which maps a region  $R$  to the number of hyperplanes in  $\mathcal{H}$  which separate  $R$  and  $B$ . Now, for two regions  $R, R' \in \mathcal{F}$ , write  $R \triangleleft R'$  iff  $R$  and  $R'$  are separated by exactly one hyperplane in  $\mathcal{H}$  and  $\ell(R) = \ell(R') - 1$ . The poset of regions  $P_{\mathcal{H}}$  is the partial order on  $\mathcal{F}$  whose covering relation is  $\triangleleft$ . It was first studied by Edelman [5].

From the point of view of this paper, the most important property of  $P_{\mathcal{H}}$  is the following.

**Theorem 4.1** (Theorem 4.3.3 in [2]). *Any linear extension of  $P_{\mathcal{H}}$  is a shelling order for  $\Delta_{\mathcal{H}}$ .*

We are now ready to state and prove the main result of this section.

**Theorem 4.2.** *If  $\mathcal{A}$  consists of hyperplanes, then  $\Delta_{\mathcal{A}, \mathcal{H}}$  is shellable.*

**Proof.** We proceed by induction over  $|\mathcal{A}|$ . When  $\mathcal{A} = \{A\}$ , we may apply Theorem 4.1 since  $\Delta_{\mathcal{A}, \mathcal{H}} = \Delta_{\mathcal{H}/A}$  in this case.

Now suppose  $|\mathcal{A}| \geq 2$  and that we have a shelling order for  $\Delta_{\mathcal{A} \setminus A, \mathcal{H}}$  for some  $A \in \mathcal{A}$ . We will append the remaining facets to this order.

The remaining facets are the facets of  $\Delta_{\{\mathcal{A}\}, \mathcal{H}} = \Delta_{\mathcal{H}/A}$ . They are divided into equivalence classes in the following way:  $F$  and  $G$  belong to the same class iff their interiors belong to the same connected component of  $\mathbb{R}^n \setminus \cup(\mathcal{A} \setminus A)$  (or, equivalently, to the same connected component of  $A \setminus \cup(\mathcal{A}/A)$ ). Observe that if  $F$  and  $G$  belong to different classes, then  $F \cap G \in \Delta_{\mathcal{A} \setminus A, \mathcal{H}}$ . Thus, it is enough to show that the facets in any equivalence class can be appended to the shelling order for  $\Delta_{\mathcal{A} \setminus A, \mathcal{H}}$ .

Without loss of generality, consider the class which contains the maximal element in  $P_{\mathcal{H}/A}$ , i.e. the region opposite to the base region. Call this class  $C$ . If  $F \in C$  and  $G \notin C$  for  $F, G \in P_{\mathcal{H}/A}$ , then some hyperplane in  $\mathcal{A}/A \subseteq \mathcal{H}/A$  separates  $F$  from  $G$ , and  $G$  is on the positive side of this hyperplane. Thus,  $F \not\leq G$ . This shows that  $C$  is an order filter in  $P_{\mathcal{H}/A}$ . According to [Theorem 4.1](#),  $\Delta_{\mathcal{H}/A}$  has a shelling order which ends with the facets in  $C$ . Now observe that  $(\cup C) \cap (\cup(P_{\mathcal{H}/A} \setminus C)) = (\cup C) \cap \Delta_{\mathcal{A} \setminus A, \mathcal{H}}$ . The facets in  $C$  may therefore be appended in this order to the shelling order for  $\Delta_{\mathcal{A} \setminus A, \mathcal{H}}$ .  $\square$

## 5. The $h$ -polynomial of $\Delta_{\mathcal{A}, \mathcal{H}}$

For brevity we write  $h(\mathcal{A}, \mathcal{H}; x)$  meaning  $h(\Delta_{\mathcal{A}, \mathcal{H}}; x)$  and similarly for  $\bar{h}$ . The following result of Steingrímsson serves as a motivating example for this section:

**Theorem 5.1** (Theorem 13 in [10]). Recall the complex  $\Delta_{\widehat{G}, \mathcal{S}_n}$  defined in [Example 3.1](#). We have

$$\frac{x\bar{h}(\widehat{G}, \mathcal{S}_n; x)}{(1-x)^n} = \sum_{m \geq 0} (m^n - P_G(m))x^m,$$

where  $P_G$  is the chromatic polynomial of  $G$ .

This theorem is interesting because of the connection between the left hand side and the Hilbert series of the Stanley–Reisner ring  $k[\Delta_{\widehat{G}, \mathcal{S}_n}]$ . In [3], Brenti began a systematic study of which polynomials arise as Hilbert polynomials of standard graded algebras. A question left open in [3], and later answered affirmatively by Almkvist [1], was whether chromatic polynomials of graphs have this property. [Theorem 5.1](#) implies something similar, namely that  $(m+1)^n - P_G(m+1)$  is the Hilbert polynomial (in  $m$ ) of a standard graded algebra; for details, see [Corollary 5.7](#).

It is well known that  $P_G(x) = x\chi(\widehat{G}; x)$ ; one way to prove it is to compare [Theorem 2.1](#) with the standard deletion–contraction recurrence for  $P_G$ . The identity suggests the possibility of extending [Theorem 5.1](#) to other complexes  $\Delta_{\mathcal{A}, \mathcal{H}}$ . This turns out to be possible at least if  $\mathcal{H} \in \{\mathcal{S}_n, \mathcal{B}_n\}$  and is the topic of this section.

Given a subspace  $T$  of  $\mathbb{R}^n$ , let  $d(T)$  denote its dimension. For a subspace arrangement  $\mathcal{T}$ , we also write

$$d(\mathcal{T}) = \max_{T \in \mathcal{T}} d(T).$$

**Lemma 5.2.** Let  $A \in \mathcal{A}$ . Then,

$$\begin{aligned} h(\mathcal{A}, \mathcal{H}; x) &= (x-1)^{d(\mathcal{A})-d(\mathcal{A} \setminus A)} h(\mathcal{A} \setminus A, \mathcal{H}; x) \\ &\quad + (x-1)^{d(\mathcal{A})-d(A)} h(\{A\}, \mathcal{H}; x) \\ &\quad - (x-1)^{d(\mathcal{A})-d(\mathcal{A}/A)} h(\mathcal{A}/A, \mathcal{H}/A; x). \end{aligned}$$

**Proof.** Each simplex in  $\Delta_{\mathcal{A}, \mathcal{H}}$  belongs to  $\Delta_{\mathcal{A} \setminus A, \mathcal{H}}$  or to  $\Delta_{\{A\}, \mathcal{H}}$  or to both. Also, observe that  $\Delta_{\mathcal{A} \setminus A, \mathcal{H}} \cap \Delta_{\{A\}, \mathcal{H}} = \Delta_{\mathcal{A}/A, \mathcal{H}/A}$ . Denoting by  $f_i(\mathcal{S}, \mathcal{T})$  the number of  $i$ -dimensional simplices in  $\Delta_{\mathcal{S}, \mathcal{T}}$ , we thus obtain for all  $i$

$$f_i(\mathcal{A}, \mathcal{H}) = f_i(\mathcal{A} \setminus A, \mathcal{H}) + f_i(\{A\}, \mathcal{H}) - f_i(\mathcal{A}/A, \mathcal{H}/A).$$

The lemma now follows from the fact that  $\dim(\Delta_{\mathcal{S}, \mathcal{T}}) = d(\mathcal{S}) - 1$ .  $\square$

We may use Lemma 5.2 to recursively compute  $h(\mathcal{A}, \mathcal{H}; x)$ . As it turns out, this recursion is particularly useful when  $\mathcal{H} \in \{\mathcal{S}_n, \mathcal{B}_n\}$ . The reason is given by the following two lemmata.

**Lemma 5.3.** *We have*

$$\frac{x\bar{h}(\Delta_{\mathcal{S}_n}; x)}{(1-x)^{n+1}} = \sum_{m \geq 0} m^n x^m$$

and

$$\frac{\bar{h}(\Delta_{\mathcal{B}_n}; x)}{(1-x)^{n+1}} = \sum_{m \geq 0} (2m+1)^n x^m.$$

**Proof.** The complexes  $\Delta_{\mathcal{S}_n}$  and  $\Delta_{\mathcal{B}_n}$  coincide with the Coxeter complexes of types  $A_{n-1}$  and  $B_n$ , respectively. For the  $h$ -polynomials this implies that  $x\bar{h}(\Delta_{\mathcal{S}_n}; x) = A_n(x)$  and  $\bar{h}(\Delta_{\mathcal{B}_n}; x) = B_n(x)$ , where  $A_n$  is the  $n$ th Eulerian polynomial and  $B_n$  is the  $n$ th  $B$ -Eulerian polynomial; see [4]. The assertions are well-known properties of these polynomials [4, Theorem 3.4.ii].  $\square$

**Lemma 5.4.**

(i) *For any subspace  $A \in L_{\mathcal{S}_n}$ , we have*

$$\frac{x\bar{h}(\{A\}, \mathcal{S}_n; x)}{(1-x)^{d(A)+2}} = \sum_{m \geq 0} m^{d(A)+1} x^m.$$

(ii) *For any subspace  $\mathcal{A} \in L_{\mathcal{B}_n}$ , we have*

$$\frac{\bar{h}(\{A\}, \mathcal{B}_n; x)}{(1-x)^{d(A)+1}} = \sum_{m \geq 0} (2m+1)^{d(A)} x^m.$$

**Proof.** A key property of  $\mathcal{S}_n$  ( $\mathcal{B}_n$ ), which is readily checked, is that its restriction to any subspace in the intersection lattice is again a type  $A$  ( $B$ ) hyperplane arrangement. Thus,  $\Delta_{\{A\}, \mathcal{S}_n} = \Delta_{\mathcal{S}_n/A} \cong \Delta_{\mathcal{S}_{d(A)+1}}$  ( $\Delta_{\{A\}, \mathcal{B}_n} = \Delta_{\mathcal{B}_n/A} \cong \Delta_{\mathcal{B}_{d(A)}}$ ). The assertions now follow from Lemma 5.3.  $\square$

The leading term of  $\chi(\mathcal{A}; x)$  is always  $x^n$ , where  $n$  is the dimension of the ambient space. It is convenient to introduce the *tail*  $T(\mathcal{A}; x) = x^n - \chi(\mathcal{A}; x)$ .

When  $\mathcal{A}$  consists of hyperplanes, the following result coincides with Theorem 5.1.

**Theorem 5.5.** *Suppose  $\mathcal{A}$  is a subspace arrangement embedded in  $\mathcal{S}_n$ . Then,*

$$\frac{x\bar{h}(\mathcal{A}, \mathcal{S}_n; x)}{(1-x)^{d(\mathcal{A})+2}} = \sum_{m \geq 0} m T(\mathcal{A}; m) x^m.$$

**Proof.** We proceed by induction over  $|\mathcal{A}|$ , noting that  $|\mathcal{A} \setminus A| < |\mathcal{A}|$  and  $|\mathcal{A}/A| < |\mathcal{A}|$  for every  $A \in \mathcal{A}$ . If  $|\mathcal{A}| = 1$ , we have  $\chi(\mathcal{A}; m) = m^{n-1} - m^{d(\mathcal{A})}$ , so that  $T(\mathcal{A}; m) = m^{d(\mathcal{A})}$ , and the theorem follows from part (i) of Lemma 5.4.

Now suppose  $|\mathcal{A}| \geq 2$  and pick a subspace  $A \in \mathcal{A}$ . Using Lemma 5.2 and the induction hypothesis, we obtain

$$\begin{aligned} \frac{x^{d(\mathcal{A})+1} h\left(\mathcal{A}, \mathcal{S}_n; \frac{1}{x}\right)}{(1-x)^{d(\mathcal{A})+2}} &= \left(\frac{1-x}{x}\right)^{d(\mathcal{A})-d(\mathcal{A} \setminus A)} \frac{x^{d(\mathcal{A})+1} h\left(\mathcal{A} \setminus A, \mathcal{S}_n; \frac{1}{x}\right)}{(1-x)^{d(\mathcal{A})+2}} \\ &\quad + \left(\frac{1-x}{x}\right)^{d(\mathcal{A})-d(A)} \frac{x^{d(\mathcal{A})+1} h\left(\{A\}, \mathcal{S}_n; \frac{1}{x}\right)}{(1-x)^{d(\mathcal{A})+2}} \\ &\quad - \left(\frac{1-x}{x}\right)^{d(\mathcal{A})-d(\mathcal{A}/A)} \frac{x^{d(\mathcal{A})+1} h\left(\mathcal{A}/A, \mathcal{S}_n/A; \frac{1}{x}\right)}{(1-x)^{d(\mathcal{A})+2}} \\ &= \sum_{m \geq 0} m(m^{n-1} - \chi(\mathcal{A} \setminus A; m))x^m \\ &\quad + \sum_{m \geq 0} m(m^{n-1} - (m^{n-1} - m^{d(A)}))x^m \\ &\quad - \sum_{m \geq 0} m(m^{d(\mathcal{A})} - \chi(\mathcal{A}/A; m))x^m \\ &= \sum_{m \geq 0} m(m^{n-1} - \chi(\mathcal{A}; m))x^m, \end{aligned}$$

where the last equality follows from the deletion–restriction theorem.

For completeness, we should also check the uninteresting case  $|\mathcal{A}| = 0$  which is not covered by the above arguments. Here,  $\bar{h}(\emptyset, \mathcal{S}_n; x) = 0$  and  $T(\emptyset; x) = 0$ , and the assertion holds.  $\square$

Employing part (ii) of Lemma 5.4 instead of part (i), and keeping track of the fact that  $\mathcal{B}_n$  is an arrangement in  $\mathbb{R}^n$ , whereas  $\mathcal{S}_n$  sits in  $\mathbb{R}^{n-1}$ , the proof of Theorem 5.5 is easily adjusted to a proof of the next result.

**Theorem 5.6.** Suppose  $\mathcal{A}$  is a subspace arrangement embedded in  $\mathcal{B}_n$ . Then,

$$\frac{\bar{h}(\mathcal{A}, \mathcal{B}_n; x)}{(1-x)^{d(\mathcal{A})+1}} = \sum_{m \geq 0} T(\mathcal{A}; 2m+1)x^m.$$

For subspace arrangements covered by Theorem 5.5 or Theorem 5.6, we may now draw the promised algebraic conclusions. To this end, for a simplicial complex  $\Gamma$  and a subcomplex  $\Gamma' \subseteq \Gamma$ , let  $\mathcal{J}_{\Gamma', \Gamma}$  be the ideal in the Stanley–Reisner ring  $k[\Gamma]$  generated by the (equivalence classes of) monomials corresponding to simplices in  $\Gamma$  that do not belong to  $\Gamma'$ .

**Corollary 5.7.** Suppose  $\mathcal{A}$  is a subspace arrangement embedded in  $\mathcal{S}_n$ . Let  $\Gamma$  denote the double cone over  $\Delta_{\mathcal{S}_n}$ , and write  $\Gamma'$  for the double cone over  $\Delta_{\mathcal{A}, \mathcal{S}_n}$  with the same cone points. The following holds:

- (i) The Hilbert polynomial of  $k[\Gamma']$  is  $F(k[\Gamma']; m) = (m+1)T(\mathcal{A}; m+1)$ .
- (ii) The Hilbert polynomial of  $\mathcal{J}_{\Gamma', \Gamma}$  is  $F(\mathcal{J}_{\Gamma', \Gamma}; m) = (m+1)\chi(\mathcal{A}; m+1)$ .



**Proof.** The dimension of  $\Gamma'$  is  $d(\mathcal{A}) + 1$ . Taking a cone over a simplicial complex does not affect the  $\bar{h}$ -polynomial. Thus,

$$\text{Hilb}(k[\Gamma']; x) = \frac{\bar{h}(\mathcal{A}, \mathcal{S}_n; x)}{(1-x)^{d(\mathcal{A})+2}} = \frac{1}{x} \sum_{m \geq 0} mT(\mathcal{A}; m)x^m,$$

where the second equality follows from [Theorem 5.5](#). This proves (i).

For (ii), we use that

$$k[\Gamma'] \cong k[\Gamma]/\mathcal{J}_{\Gamma', \Gamma}.$$

For the Hilbert series, this implies

$$\text{Hilb}(k[\Gamma']; x) = \text{Hilb}(k[\Gamma]; x) - \text{Hilb}(\mathcal{J}_{\Gamma', \Gamma}; x).$$

From part (i) and the fact that

$$\text{Hilb}(k[\Gamma]) = \frac{\bar{h}(\Delta_{\mathcal{S}_n}; x)}{(1-x)^{n+1}} = \frac{1}{x} \sum_{m \geq 0} m^n x^m,$$

we conclude

$$\text{Hilb}(\mathcal{J}_{\Gamma', \Gamma}; x) = \frac{1}{x} \sum_{m \geq 0} m^n x^m - \frac{1}{x} \sum_{m \geq 0} mT(\mathcal{A}; m)x^m = \frac{1}{x} \sum_{m \geq 0} m\chi(\mathcal{A}; m)x^m. \quad \square$$

The situation for  $\mathcal{B}_n$  is analogous, although we use cones instead of double cones. This is a manifestation of the fact that  $\mathcal{B}_n$  and  $\mathcal{S}_n$  differ by one in dimension.

**Corollary 5.8.** *Suppose  $\mathcal{A}$  is a subspace arrangement embedded in  $\mathcal{B}_n$ . Let  $\Gamma$  denote the cone over  $\Delta_{\mathcal{B}_n}$ , and write  $\Gamma'$  for the cone over  $\Delta_{\mathcal{A}, \mathcal{B}_n}$  with the same cone point. Then, the following holds:*

- (i) *The Hilbert polynomial of  $k[\Gamma']$  is  $F(k[\Gamma']; m) = T(\mathcal{A}; 2m + 1)$ .*
- (ii) *The Hilbert polynomial of  $\mathcal{J}_{\Gamma', \Gamma}$  is  $F(\mathcal{J}_{\Gamma', \Gamma}; m) = \chi(\mathcal{A}; 2m + 1)$ .*

**Proof.** Proceeding as in the proof of [Corollary 5.7](#), using [Theorem 5.6](#) instead of [Theorem 5.5](#), we prove (i) by observing

$$\text{Hilb}(k[\Gamma']; x) = \frac{\bar{h}(\mathcal{A}, \mathcal{B}_n; x)}{(1-x)^{d(\mathcal{A})+1}} = \sum_{m \geq 0} T(\mathcal{A}; 2m + 1)x^m.$$

For (ii), note that

$$\text{Hilb}(k[\Gamma]; x) = \frac{\bar{h}(\Delta_{\mathcal{B}_n}; x)}{(1-x)^{n+1}} = \sum_{m \geq 0} (2m + 1)^n x^m.$$

Thus,

$$\text{Hilb}(\mathcal{J}_{\Gamma', \Gamma}; x) = \sum_{m \geq 0} (2m + 1)^n x^m - \sum_{m \geq 0} T(\mathcal{A}; 2m + 1)x^m = \sum_{m \geq 0} \chi(\mathcal{A}; 2m + 1)x^m. \quad \square$$

Any hypergraph (without inclusions among edges)  $G$  on  $n$  vertices corresponds to a subspace arrangement  $\widehat{G}$  embeddable in  $S_n$ . The construction is virtually the same as in [Example 3.1](#); with the hyperedge  $\{i_1, \dots, i_t\}$  is associated the subspace given by  $x_{i_1} = \dots = x_{i_t}$ . As for ordinary graphs (the hyperplane case), we have  $x\chi(\widehat{G}; x) = P_G(x)$ ; cf. [[9](#), Theorem 3.4]. In this way, [Corollary 5.7](#) allows us to interpret chromatic polynomials of hypergraphs in terms of Hilbert polynomials. For ordinary graphs, this is the content of Steingrímsson's [[10](#), Corollary 10].

[Corollary 5.8](#), too, has an impact on chromatic polynomials. Any *signed graph* (in the sense of Zaslavsky [[11](#)])  $G$  on  $n$  vertices corresponds to a hyperplane arrangement  $\widehat{G} \subseteq \mathcal{B}_n$ , and vice versa. A signed graph  $G$  has a chromatic polynomial  $P_G(x)$ , and  $P_G(x) = \chi(\widehat{G}; x)$  [[12](#)].

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